

Cosmic Censorship Conjecture revisited: Covariantly

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In this paper we study the dynamics of the trapped region using a frame independent semi-tetrad covariant formalism for general Locally Rotationally Symmetric (LRS) class II spacetimes. We covariantly prove some important geometrical results for the apparent horizon, and state the necessary and sufficient conditions for a singularity to be locally naked. These conditions bring out, for the first time in a quantitative and transparent manner, the importance of the Weyl curvature in deforming and delaying the trapped region during continual gravitational collapse, making the central singularity locally visible.

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I. INTRODUCTION

Since Penrose proposed the famous *Cosmic Censorship Conjecture (CCC)* in 1969 [1], stating that singularities observable from the outside will never arise in generic gravitational collapse which starts from a perfectly reasonable nonsingular initial state, there have been numerous attempts towards validating this conjecture by means of a rigorous mathematical proof. However, this conjecture remains unproved, and it has been recognised as one of the most important open problems in gravitational physics. The key point here is that the validity of this conjecture will confirm the already widely accepted and applied theory of black hole dynamics, which has considerable amount of astrophysical applications. On the other hand, its overturn will throw the black hole dynamics into serious doubt. This is because most of the important fundamental global theorems in black hole physics assume that the spacetime manifold is *future asymptotically predictable*. In other words this condition ensures that there should be no singularity to the future of the partial Cauchy surface which is ‘naked’ or visible from the future null infinity [2].

Although no conclusive proof or disproof of CCC could be formulated, the quest gave rise to a number of counter examples which showed there are shell focusing naked singularities occurring at the centre of spherically symmetric dust, perfect fluids or radiation shells (see for example [3, 4] and the references therein). We can, in principle, rule out these naked singularities by stating that dust or perfect fluids are not really ‘fundamental’ forms of matter field, as their properties are not derived

from a ‘proper’ Lagrangian. However, if the cosmic censorship is to be established as a rigorous mathematical theorem, this objection has to be made precise in terms of a clear and simple restriction on the energy momentum tensor of the matter field. This is necessary because in the above mentioned examples, the matter satisfies physically reasonable conditions such as the energy conditions or a well posed initial value formulation for the Einstein field equations. Also, these forms of matter are widely used in discussing astrophysical processes, such as collapsing stars.

Extensive studies of various dynamical collapse models for a wide range of matter fields, mainly spherically symmetric, continued over the past two decades, investigating the final outcome of gravitational collapse (refer to [5] for a detailed analysis on this subject). The generic conclusions which emerged from these studies were extraordinary as they conclusively indicated that while the collapse always produces curvature generated fireballs characterised by diverging densities and curvatures, trapped surfaces may not develop early enough to always shield this process from an outside observer. Not just isolated trajectories but families of non-spacelike geodesics emerge from such a naked singularity, providing a non-zero measure set of trajectories escaping away.

An obvious question of considerable interest and significance, is then the following: What are the possible physical and geometrical factors that are responsible for this delay in the formation of trapped regions, that cover the spacetime singularity? In other words, we would like to inquire about the physical and geometrical effects operating during the continual collapse of a massive matter cloud that lead to the formation of a locally naked singularity rather than a black hole, or vice versa. Such an investigation should help us in obtaining a better understanding of the physics of black hole or naked singularity formation in gravitational collapse. Towards this end, the pioneering work was done by Joshi et al [6] followed

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by [7], which showed that spacetime shear plays a crucial role in determining the end state of continual collapse. The important insights that emerged from these investigations were that there exists a remarkable connection between spacetime shear and inhomogeneity of collapsing matter cloud that can distort the geometry of the trapped region in such a way that the central singularity can be locally naked.

We continue with this investigation to obtain more transparent physical picture of the problem; in this paper we study the dynamics of the trapped region using the frame independent semi-tetrad covariant formalism for general Locally Rotationally Symmetric (LRS) class II spacetimes (of which spherical symmetry is a subclass) [8]. We write down the field equations for the LRS II spacetimes as propagation, evolution and constraint equations in terms of different covariant scalars that have well defined physical and geometrical interpretations. We deduce the equations of null geodesics for these spacetimes in terms of these scalars, and we find the equation of the apparent horizon (the boundary of the trapped region) where the expansion of the null geodesics vanishes. We covariantly prove some geometric results for the apparent horizon and state the necessary and sufficient conditions for a singularity to be locally naked. These conditions bring out for the first time, in a quantitative and transparent manner, the importance of the Weyl curvature in deforming and delaying the trapped region to make the central singularity locally naked.

As we know the Weyl tensor, which is the trace-free part of the Riemann curvature tensor, gives the measure of the ‘pure’ geometrical effect on the spacetime manifold, as this tensor can be non-zero even in the absence of any matter field. The Weyl tensor depicts the tidal forces experienced by the test particles, resulting in volume distortion and generating the spacetime shear. It also gives a measure of gravitational wave propagation. Four dimensional spacetimes with vanishing Weyl tensor are conformally flat. We rigorously show that for such spacetimes, a collapsing perfect fluid necessarily ends up in a black hole end state as the singularity is always hidden within the horizon. This then relates conformal flatness with local visibility (or otherwise) of a spacetime singularity.

The paper is organised as follows: In the next section we provide a brief description of the semi-tetrad 1+3 and 1+1+2 formalisms, and define the covariant kinematical and dynamical variables that have well defined geometrical and physical significance. In section 3, we use these variables to write down the field equations for LRS-II spacetimes. In section 4, we deduce the equations of null geodesics and define the expressions for expansion, shear etc., for the null congruence in the two dimensional null screen space. In section 5, we derive the equation for apparent horizon, which is the boundary of the trapped region in terms of these covariant variables. This then gives a local frame independent description of the hori-

zon, and we prove some important covariant results for spherical collapsing shells crossing the horizon (*i.e.* getting trapped). In section 6, we give the necessary and sufficient conditions for a spacetime singularity to be locally naked. Finally in the last section we use this result to establish the nature of the singularity which develops as the end state of gravitational collapse, for some special cases.

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this paper, Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $(-, +, +, +)$ signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \quad (1)$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (*i.e.* symmetric in the lower indices), defined by

$$\Gamma^a{}_{bd} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \quad (2)$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd}R_{cadb}. \quad (3)$$

The symmetrization and the antisymmetrization over the indexes of a tensor are defined as

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \quad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}). \quad (4)$$

The Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} [R - 2\Lambda - 2\mathcal{L}_m], \quad (5)$$

variation of which gives the Einstein’s field equations as

$$G_{ab} + \Lambda g_{ab} = T_{ab}. \quad (6)$$

II. SEMI-TETRAD COVARIANT FORMALISMS

Spacetimes can be described using tetrad formalisms or metric (or coordinate) based approaches. The tetrad formalisms range from the Newman-Penrose null tetrad method, 3+1 ADM decomposition, 1+3 covariant approach developed by Ehlers and Ellis to 1+1+2 covariant formalism. These include either a full tetrad approach or a ‘partial’ covariant approach where only one or two tetrad vectors are chosen. In this section we give a brief review of the last two formalisms mentioned above.

A. 1+3 Covariant formalism

This formalism [9] is based on a local 1+3 threading of the spacetime manifold with respect to a timelike congruence, such that spacetime is locally decomposed into

space and time. The 1+3 formalism has been a useful tool for understanding many geometrical and physical aspects of relativistic fluid flows, both in non-linear GR studies or in the gauge invariant, covariant perturbation formalism [10].

In this approach we must first define a time-like congruence with a unit tangent vector u^a . The natural choice of this vector in our case will be the tangent to the matter flow lines. Then the spacetime is split locally in the form $R \otimes V$ where R denotes the worldline along u^a and V is the 3-space perpendicular to u^a . Any vector X^a can then be projected on the 3-space by the projection tensor $h^a_b = g^a_b + u^a u_b$. The choice of the timelike vector naturally defines two derivatives: the vector u^a is used to define the *covariant time derivative* along the observers' worldlines (denoted by a dot) for any tensor $S^{a..b}_{c..d}$, given by

$$\dot{S}^{a..b}_{c..d} = u^e \nabla_e S^{a..b}_{c..d} \quad (7)$$

and the tensor h_{ab} is used to define the fully orthogonally *projected covariant derivative* D for any tensor $S^{a..b}_{c..d}$:

$$D_e S^{a..b}_{c..d} = h^a_f h^p_{c..} h^b_g h^q_d h^r_e \nabla_r S^{f..g}_{p..q}, \quad (8)$$

with total projection on all the free indices. Angle brackets denote orthogonal projections of vectors, and the orthogonally *projected symmetric trace-free* PSTF part of tensors:

$$V^{(a)} = h^a_b V^b, \quad S^{(ab)} = \left[h^{(a}_c h^b)_{d} - \frac{1}{3} h^{ab} h_{cd} \right] S^{cd}. \quad (9)$$

This splitting of spacetime also naturally defines the 3-volume element

$$\epsilon_{abc} = -\sqrt{|g|} \delta^0_{[a} \delta^1_b \delta^2_c \delta^3_{d]} u^d, \quad (10)$$

with the following identities

$$\epsilon_{abc} \epsilon^{def} = 3! h^d_{[a} h^e_b h^f_{c]}; \quad \epsilon_{abc} \epsilon^{dec} = 2! h^d_{[a} h^e_{b]}. \quad (11)$$

The covariant derivative of the time-like vector u^a can now be decomposed into the irreducible part as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \epsilon_{abc} \omega^c, \quad (12)$$

where $A_a = \dot{u}_a$ is the acceleration, $\Theta = D_a u^a$ is the expansion, $\sigma_{ab} = D_{(a} u_{b)}$ is the shear tensor and $w^a = \epsilon^{abc} D_b u_c$ is the vorticity vector. Similarly the Weyl curvature tensor can be decomposed irreducibly into the Gravito-Electric and Gravito-Magnetic parts as

$$E_{ab} = C_{abcd} u^c u^d = E_{(ab)}; \quad H_{ab} = \frac{1}{2} \epsilon_{acd} C^{cd}_{be} u^e = H_{(ab)}, \quad (13)$$

which allows for a covariant description of tidal forces and gravitational radiation. The energy momentum tensor for a general matter field can be similarly decomposed as follows:

$$T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}, \quad (14)$$

where $\mu = T_{ab} u^a u^b$ is the energy density, $p = (1/3) h^{ab} T_{ab}$ is the isotropic pressure, $q_a = q_{(a)} = -h^c_a T_{cd} u^d$ is the 3-vector defining the heat flux and $\pi_{ab} = \pi_{(ab)}$ is the anisotropic stress.

B. 1+1+2 Covariant formalism

A natural extension to the 1+3 formalism, which is optimized for spacetimes having a preferred spatial direction (for example spherical symmetry), is the 1+1+2 formalism developed recently by Clarkson and Barrett and it has been used extensively to study perturbations of black holes [11–13]. In this formalism we first proceed with the same split of the 1+3 approach followed by another split along a preferred spatial direction. This allows us to derive a set of covariant scalar variables which are more advantageous to treat systems with one preferred direction. For example in spherically symmetric systems the governing field equations in the 1+1+2 approach are scalar equations and are much simpler than the ones of the 1+3 formalism which are in general tensorial.

Hence in this approach we choose a further preferred vector field e^a which performs additional slicing of the ‘3-space’. This new vector field has to be orthogonal to u^a such that it satisfies $e^a e_a = 1$, $u^a e_a = 0$. The 1+3 projection tensor $h^a_b \equiv g^a_b + u_a u^b$ combined with e^a defines a new projection tensor N_{ab} ,

$$N_a^b \equiv h_a^b - e_a e^b = g_a^b + u_a u^b - e_a e^b, \quad (15)$$

which projects vectors orthogonal to e^a and u^a ($e^a N_{ab} = 0 = u^a N_{ab}$) onto a 2-surface which is defined as the sheet ($N_a^a = 2$). The volume element of this 2-surface is then Levi-Civita 2-tensor, derived from the volume element ϵ_{abc} for the observers' rest spaces by

$$\varepsilon_{ab} \equiv \epsilon_{abc} e^c = u^d \eta_{dabc} e^c; \quad \varepsilon_{ab} e^b = 0 = \varepsilon_{(ab)}. \quad (16)$$

Any 3-vector ψ^a can now be irreducibly split into a scalar, Ψ , which is the part of the vector parallel to e^a , and a vector, Ψ^a , lying in the 2-surface orthogonal to e^a :

$$\psi^a = \Psi e^a + \Psi^a, \quad \text{where } \Psi \equiv \psi_a e^a, \quad \text{and } \Psi^a \equiv N^{ab} \psi_b \equiv \psi^{\bar{a}}, \quad (17)$$

where the bar over the index denotes projection with N_{ab} . Similarly, we can do the same for any tensor, ψ_{ab} , as follows:

$$\psi_{ab} = \psi_{(ab)} = \Psi (e_a e_b - \frac{1}{2} N_{ab}) + 2\Psi_{(a} e_{b)} + \Psi_{ab}, \quad (18)$$

where

$$\begin{aligned} \Psi &\equiv e^a e^b \psi_{ab} = -N^{ab} \psi_{ab}, \\ \Psi_a &\equiv N_a^b e^c \psi_{bc} = \Psi_{\bar{a}}, \\ \Psi_{ab} &\equiv \left(N_{(a}^c N_{b)}^d - \frac{1}{2} N_{ab} N^{cd} \right) \psi_{cd} \equiv \Psi_{\{ab\}}. \end{aligned} \quad (19)$$

The curly brackets denote the PSTF tensors on the 2-sheets. Apart from the ‘time’ (dot) derivative, of an object, we now introduce two new derivatives, which e^a defines, for any object $\psi_{a...b}^{c...d}$:

$$\hat{\psi}_{a..b}^{c..d} \equiv e^f D_f \psi_{a..b}^{c..d}, \quad (20)$$

$$\delta_f \psi_{a..b}^{c..d} \equiv N_a^f \dots N_b^g N_h^c \dots N_i^d N_f^j D_j \psi_{f..g}^{i..j}. \quad (21)$$

The hat-derivative is the derivative along the e^a vector-field in the surfaces orthogonal to u^a . The δ -derivative is the projected derivative onto the sheet, with the projection on every free index.

We can now split the usual 1+3 kinematical and Weyl quantities into the irreducible set $\{\theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}$ using (17) and (18) as follows [13]:

$$\dot{u}^a = \mathcal{A}e^a + \mathcal{A}^a, \quad (22)$$

$$\omega^a = \Omega e^a + \Omega^a, \quad (23)$$

$$\sigma_{ab} = \Sigma(e_a e_b - \frac{1}{2} N_{ab}) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}, \quad (24)$$

$$E_{ab} = \mathcal{E}(e_a e_b - \frac{1}{2} N_{ab}) + 2\mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \quad (25)$$

$$H_{ab} = \mathcal{H}(e_a e_b - \frac{1}{2} N_{ab}) + 2\mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}. \quad (26)$$

The shear scalar, σ , for example, may be expressed in the form

$$\sigma^2 \equiv \frac{1}{2} \sigma_{ab} \sigma^{ab} = \frac{3}{4} \Sigma^2 + \Sigma_a \Sigma^a + \frac{1}{2} \Sigma_{ab} \Sigma^{ab}. \quad (27)$$

Similarly we may split the fluid variables q^a and π_{ab} ,

$$q^a = Qe^a + Q^a, \quad (28)$$

$$\pi_{ab} = \Pi(e_a e_b - \frac{1}{2} N_{ab}) + 2\Pi_{(a} e_{b)} + \Pi_{ab}. \quad (29)$$

We are now able to decompose the covariant derivative of e^a in the direction orthogonal to u^a into its irreducible parts giving

$$D_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} + \zeta_{ab}, \quad (30)$$

where

$$a_a \equiv e^c D_c e_a = \hat{e}_a, \quad (31)$$

$$\phi \equiv \delta_a e^a, \quad (32)$$

$$\xi \equiv \frac{1}{2} \epsilon^{ab} \delta_a e_b, \quad (33)$$

$$\zeta_{ab} \equiv \delta_{\{a} e_{b\}}. \quad (34)$$

We see that on the 3-space, moving along the preferred vector e^a , ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear of e^a* (i.e. the distortion of the sheet) and a^a its *acceleration*. We can also interpret ξ as the *vorticity* associated with e^a so that it is a representation of the ‘‘twisting’’ or rotation of the sheet.

The Ricci identities for e_a is given by:

$$R_{abc} \equiv 2\nabla_{[a} \nabla_{b]} e_c - R_{abcd} e^d = 0, \quad (35)$$

where R_{abcd} is the Riemann curvature tensor. And the full covariant derivative of e_a and u_a is now written as:

$$\begin{aligned} \nabla_a e_b &= -\mathcal{A}u_a u_b - u_a \alpha_b + (\Sigma + \frac{1}{3}\theta) e_a u_b + \xi \varepsilon_{ab} \\ &\quad + (\Sigma_a - \varepsilon_{ac} \Omega^c) u_b + e_a a_b + \frac{1}{2} \phi N_{ab}, \end{aligned} \quad (36)$$

$$\begin{aligned} \nabla_a u_b &= -u_a (\mathcal{A}e_b + \mathcal{A}_b) + e_a e_b (\frac{1}{3}\theta + \Sigma) + \Omega \varepsilon_{ab} \\ &\quad + e_a (\Sigma_b + \varepsilon_{bc} \Omega^c) + (\Sigma_a - \varepsilon_{ac} \Omega^c) e_b \\ &\quad + N_{ab} (\frac{1}{3}\theta - \frac{1}{2}\Sigma) + \Sigma_{ab}, \end{aligned} \quad (37)$$

We also write one more useful relation

$$\hat{u}_a = (\frac{1}{3}\theta + \Sigma) e_a + \Sigma_a + \varepsilon_{ab} \Omega^b. \quad (38)$$

III. LRS-II SPACETIMES

A spacetime manifold (\mathcal{M}, g) is called *locally isotropic*, if every point $p \in (\mathcal{M}, g)$ has continuous non-trivial isotropy group. When this group consists of spatial rotations the spacetime is called *locally rotationally symmetric* (LRS) [8]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction, covariantly defined (for example, by the vorticity vector field or a non-vanishing acceleration of the matter fluids, etc.). The 1+1+2 formalism is therefore ideally suited for covariant description of these spacetimes. The preferred spatial direction in the LRS spacetimes constitutes a local axis of symmetry and in this case e^a is just a vector pointing along the axis of symmetry. Since LRS spacetimes are isotropic about this axis, *all* 2-vectors and 2-tensors vanish, so that there are no preferred directions in the sheet. Thus, all the non-zero 1+1+2 variables are covariantly defined scalars. The variables $\{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}$, fully describe LRS spacetimes, and are what is solved for in the 1+1+2 approach.

Within the LRS cases is the LRS-II class that admits spherically symmetric solutions and is free of rotation, thus allowing for the vanishing of the variables Ω , ξ and \mathcal{H} . The set of quantities that fully describe LRS class II spacetimes are $\{\mathcal{A}, \Theta, \phi, \Sigma, \mathcal{E}, \mu, p, \Pi, Q\}$. It was shown that the most general metric for LRS II can be written as [14]

$$\begin{aligned} ds^2 &= -A^{-2}(t, \chi) dt^2 + B^2(t, \chi) d\chi^2 \\ &\quad + C^2(t, \chi) [dy^2 + D^2(y, k) dz^2], \end{aligned} \quad (39)$$

where t and χ are the affine parameters along the integral curves of u^a and e^a respectively, and $k = (1, 0, -1)$ describes the closed, flat or open geometry of the 2-sheets respectively.

We now have all the tools to derive the propagation and the evolution equations for the LRS-II variables (for more details see [13]). These equations are obtained by the Ricci identities of the vectors u^a and e^a and the doubly contracted Bianchi identities.

Propagation:

$$\begin{aligned}\hat{\phi} &= -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\theta + \Sigma\right) \left(\frac{2}{3}\theta - \Sigma\right) \\ &\quad - \frac{2}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi, \quad (40)\end{aligned}$$

$$\hat{\Sigma} - \frac{2}{3}\hat{\theta} = -\frac{3}{2}\phi\Sigma - Q, \quad (41)$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\theta\right)Q. \quad (42)$$

Evolution:

$$\dot{\phi} = -\left(\Sigma - \frac{2}{3}\theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + Q, \quad (43)$$

$$\begin{aligned}\dot{\Sigma} - \frac{2}{3}\dot{\theta} &= -\mathcal{A}\phi + 2\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2 \\ &\quad + \frac{1}{3}(\mu + 3p - 2\Lambda) - \mathcal{E} + \frac{1}{2}\Pi, \quad (44)\end{aligned}$$

$$\begin{aligned}\dot{\mathcal{E}} - \frac{1}{3}\dot{\mu} + \frac{1}{2}\dot{\Pi} &= +\left(\frac{3}{2}\Sigma - \theta\right)\mathcal{E} + \frac{1}{4}\left(\Sigma - \frac{2}{3}\theta\right)\Pi \\ &\quad + \frac{1}{2}\phi Q - \frac{1}{2}(\mu + p)\left(\Sigma - \frac{2}{3}\theta\right). \quad (45)\end{aligned}$$

Propagation/evolution:

$$\begin{aligned}\hat{\mathcal{A}} - \dot{\theta} &= -(\mathcal{A} + \phi)\mathcal{A} + \frac{1}{3}\theta^2 + \frac{3}{2}\Sigma^2 \\ &\quad + \frac{1}{2}(\mu + 3p - 2\Lambda), \quad (46)\end{aligned}$$

$$\dot{\mu} + \hat{Q} = -\theta(\mu + p) - (\phi + 2\mathcal{A})Q - \frac{3}{2}\Sigma\Pi, \quad (47)$$

$$\begin{aligned}\dot{Q} + \hat{p} + \hat{\Pi} &= -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\frac{4}{3}\theta + \Sigma\right)Q \\ &\quad - (\mu + p)\mathcal{A}. \quad (48)\end{aligned}$$

The 3-Ricci scalar of the spacelike 3-space orthogonal to u^a can be expressed as

$${}^3R = -2\left[\hat{\phi} + \frac{3}{4}\phi^2 - K\right], \quad (49)$$

where K is the Gaussian curvature of the 2-sheet defined by ${}^2R_{ab} = KN_{ab}$. In terms of the covariant scalars we can write the Gaussian curvature K as

$$K = \frac{1}{3}(\mu + \Lambda) - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2 - \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right)^2. \quad (50)$$

Finally the evolution and propagation equations for the Gaussian curvature K are

$$\dot{K} = -\left(\frac{2}{3}\theta - \Sigma\right)K, \quad (51)$$

$$\hat{K} = -\phi K. \quad (52)$$

IV. NULL GEODESICS IN LRS-II SPACETIMES

In this section, we derive the equation for null geodesics in LRS-II spacetimes and investigate the geometry of these null congruences. Null geodesics (light rays) are characterised by the curves $x^a(\nu)$ on (\mathcal{M}, g) , where ν is an affine parameter along the geodesics. The tangent to these curves is defined by

$$k^a = \frac{dx^a}{d\nu}(\nu) \quad (53)$$

where k^a is a null vector obeying

$$k^a k_a = 0. \quad (54)$$

Also, since the tangent vector to the geodesic is parallelly propagated to itself, we can write

$$k^b \nabla_b k^a = \frac{\delta k^a}{\delta \nu} = 0, \quad (55)$$

where $\frac{\delta}{\delta \nu} = k^b \nabla_b$ as the derivative along the ray. In the usual 1+3 decomposition of null geodesics, we define the unit spacial vector n_a as

$$n^a n_a = 1, \quad n^a u_a = 0. \quad (56)$$

The null vector k^a can now be split in the usual way

$$k^a = E(u^a + n^a), \quad (57)$$

The 1+1+2 split of n^a can then be performed, such that [15, 16]

$$k^a = E(u^a + \kappa e^a + \kappa^a), \quad (58)$$

where $E \equiv -u_a k^a$ can be interpreted as the energy associated with the ray, $\kappa \equiv k^a e_a$ is the magnitude of the component along the preferred spatial direction, and κ^a is the component lying on the 2-sheet.

At this point let us define the notion of locally *outgoing* and *incoming* null geodesics with respect to the preferred spatial direction. Consider any open subset \mathcal{S} of (\mathcal{M}, g) and let $x^a(\nu)$ be a null geodesic in \mathcal{S} . Let k^a be the tangent to this geodesic. If $e^a k_a > 0$ in \mathcal{S} then the geodesic is considered to be outgoing with respect to the preferred direction in \mathcal{S} . Similarly $e^a k_a < 0$ denotes an incoming geodesic. This can also be explained in terms of the local co-ordinates. Let p_1 and p_2 be two points on $x^a(\nu)$ such that p_2 is in the causal future of p_1 . Both these points can be labelled by the values of ‘ t ’ and ‘ χ ’ (which are the affine parameters on the integral curves of the vectors u^a and e^a respectively) and the local coordinates on the 2-sheets. Let the values of χ at these points be χ_1 and χ_2 respectively. If $\chi_2 > \chi_1$ then the geodesic is considered to be outgoing and if $\chi_2 < \chi_1$ then the geodesic is considered to be incoming (with respect to the preferred direction).

A. The propagation equations for the null geodesics

The propagation equations for the energy E and the component κ in a LRS-II spacetime (where the sheet component κ^a vanishes) can be derived by substituting (58) into (55), and projecting the expression along the timelike direction (u^a) and along the radial direction (e^a) [15, 16]

$$\frac{\delta E}{\delta \nu} = E' = -E^2 \kappa \mathcal{A} - \frac{3}{2}\Sigma \kappa^2 E^2 - E^2 \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right), \quad (59)$$

$$\frac{\delta \kappa}{\delta \nu} = \kappa' = E(1 - \kappa^2) \left(\frac{1}{2}\phi - \mathcal{A} - \frac{3}{2}\Sigma\right). \quad (60)$$

We have used the following properties

$$\begin{aligned} k^b u_b &= -E, \quad k^b e_b = E\kappa, \quad N^a_b k^b = E\kappa^a, \\ \varepsilon^a_b k^b &= E\varepsilon^a_b \kappa^b, \quad u_a \kappa^a = 0, \quad e_a \kappa^a = 0, \end{aligned} \quad (61)$$

as well as [16]

$$\begin{aligned} u'_a &= E\mathcal{A}e_a + E\kappa\left(\frac{1}{3}\Theta + \Sigma\right)e_a \\ &\quad + E\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\kappa_a + E\Omega\varepsilon_{ab}\kappa^b, \\ e'_a &= E\mathcal{A}u_a + E\kappa\left(\Sigma + \frac{1}{3}\Theta\right)u_a + \frac{1}{2}E\phi\kappa_a + E\xi\varepsilon_{ab}\kappa^b, \end{aligned} \quad (62)$$

which are obtained from (36, 37) with the definition of prime introduced above. For null rays along the preferred spatial direction we have $\kappa = \pm 1$ (denoting the outgoing and incoming geodesics). Then we can easily see that the equation (60) is satisfied identically and (59) simplifies to

$$\frac{\delta E}{\delta \nu} = E' = \mp E^2 \mathcal{A} - E^2 \left(\Sigma + \frac{1}{3}\Theta\right). \quad (63)$$

B. The Screen-Space

As we have already seen, for LRS-II spacetimes the outgoing null vector is defined as

$$k^a = E(u^a + e^a). \quad (64)$$

Since the hypersurface orthogonal to null vector k^a , contains k^a and hence the projection onto a locally orthogonal space now has to be defined differently. Let us now define the projection tensor \tilde{h}_{ab} , which projects tensors and vectors into the 2-D screen space orthogonal to k^a , as [15]

$$\tilde{h}_{ab} \equiv g_{ab} + 2k_{(a}l_{b)}, \quad \tilde{h}_a^a = 2, \quad \tilde{h}_{ac}\tilde{h}_b^c = \tilde{h}_{ab}, \quad \tilde{h}_{ab}k^b = 0, \quad (65)$$

where l_a is null ingoing geodesic that obeys

$$l^a l_a = 0, \quad k^a l_a = -1 \text{ and } \frac{\delta l^a}{\delta \nu} = k^b \nabla_b l^a = 0. \quad (66)$$

Using these definitions, the general form of l^a can be written as:

$$l^a = \frac{1}{2E}(u^a - e^a), \quad (67)$$

and substituting (67) into (65) the screen-space projection tensor is obtained as

$$\tilde{h}_{ab} = g_{ab} + u_a u_b - e_a e_b. \quad (68)$$

It is interesting to note that although defined differently, we automatically have

$$\tilde{h}_{ab} = N_{ab} \quad (69)$$

An expression for any vector or tensor lying on the 2-D surface can be obtained by

$$\tilde{V}^a = \tilde{h}^a_b V_b, \quad \tilde{T}^{a\dots c}_{b\dots d} = \tilde{h}^a_e \tilde{h}^f_b \dots \tilde{h}^h_d T^{e\dots g}_{f\dots h}. \quad (70)$$

For completeness, we will write here the full 1+3 decomposition of the covariant derivative of the null vector k^a for a general spacetime [15]

$$\nabla_b k_a = \frac{1}{2}\tilde{h}_{ab}\tilde{\Theta}_{out} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} + \tilde{X}_a k_b + \tilde{Y}_b k_a + \lambda k_a k_b, \quad (71)$$

where

$$\tilde{X}_a = \frac{1}{E}e^d \nabla_d k_a, \quad \tilde{Y}_a = \frac{1}{E}e^d \nabla_a k_d, \quad \lambda = -\frac{1}{E^2}e^c e^d \nabla_d k_c, \quad (72)$$

and $\tilde{\Theta}_{out}$, $\tilde{\sigma}_{ab}$, $\tilde{\omega}_{ab}$ represent the expansion, shear and vorticity of the outgoing null congruence respectively. A similar decomposition can be done for the incoming null geodesic l^a .

V. APPARENT HORIZON IN SPHERICALLY SYMMETRIC SPACETIMES

As we now have a complete picture of the equations governing the geometry of null geodesics in LRS-II spacetimes, we will use these results in this section to derive some important propositions regarding the apparent (or cosmological) horizons. Henceforth we will only consider the class of spherically symmetric spacetimes which belongs to the LRS-II class with an extra condition of positivity of the Gaussian curvature of the 2-sheets ($K > 0$).

Let us briefly discuss the concept of a *closed trapped surface* for a spherically symmetric spacetime. As described in [2], we will consider a spherical emitter, surrounding a massive body, emitting a flash of light. In the normal circumstances, by Huygen's construction, there will be outgoing and incoming spherical wavefronts and the surface area of the outgoing wavefronts will be greater than the emitting sphere while that of the incoming wavefront will be less than the emitting sphere. In other words, in a normal situation, the volume expansion of the outgoing null congruence orthogonal to the sphere is always positive ($\tilde{\Theta}_{out} > 0$) while that of the incoming congruence is always negative ($\tilde{\Theta}_{in} < 0$). However, if sufficiently large amount of matter is present within the emitting sphere, the surface areas of *both* incoming and outgoing wavefronts will be less than that of the emitting sphere. The surface of the emitting sphere is then said to be a *closed trapped surface*. In other words the volume expansion of the outgoing null congruence orthogonal to a closed trapped surface is negative. The collection of all closed trapped surfaces in a four dimensional spacetime manifold constitutes a *trapped region*. The boundary of the trapped region is called the *apparent horizon* where the volume expansion of the outgoing null congruence vanishes ($\tilde{\Theta}_{out} = 0$). For expanding cosmologies (like de-Sitter universe) we can similarly define the *cosmological horizon* where ($\tilde{\Theta}_{in} = 0$). For a detailed discussion on trapped surfaces and black holes we refer to [18] (and the references therein).

Proposition 1. *For any spherically symmetric spacetime (\mathcal{M}, g) that allows a local 1+1+2 splitting, the ap-*

parent horizon is described by the curve $(\frac{2}{3}\Theta - \Sigma + \phi) = 0$, while the cosmological horizon is described by $(\frac{2}{3}\Theta - \Sigma - \phi) = 0$, in the local $[u, e]$ plane.

Proof. We know, by definition, $\tilde{\sigma}_a^a = 0$, $e^a \tilde{\sigma}_{ab} = 0 = u^a \tilde{\sigma}_{ab}$, $e^a \tilde{\omega}_{ab} = 0 = u^a \tilde{\omega}_{ab}$. Also together with the properties in (65), we can easily conclude that

$$\begin{aligned}\tilde{\Theta}_{out} &= \tilde{h}^{ab} \nabla_b k_a \\ &= EN^{ab} \nabla_a (u_b + e_b).\end{aligned}\quad (73)$$

Now using (37) and (36) in (73) we obtain,

$$\tilde{\Theta}_{out} = E \left(\frac{2}{3}\Theta - \Sigma + \phi \right). \quad (74)$$

Hence for a null congruence with non-zero energy E , $\tilde{\Theta}_{out} = 0$ implies that $(\frac{2}{3}\Theta - \Sigma + \phi) = 0$. Similarly we can use the decomposition of the incoming null vector l^a to obtain the equation for the cosmological horizon. $\tilde{\Theta}_{in} = 0$ will then imply $(\frac{2}{3}\Theta - \Sigma - \phi) = 0$. \square

Proposition 2. *For any spherically symmetric spacetime (\mathcal{M}, g) that allows a local $1+1+2$ splitting, the gradient of the Gaussian curvature of the 2-sheets that intersect with the apparent (or cosmological) horizon is null.*

Proof. Let us calculate the quantity $\nabla_a K \nabla^a K$ for a spherically symmetric spacetime (where $K \neq 0$):

$$\nabla_a K \nabla^a K = (-u^a u^b + e^a e^b) \nabla_a K \nabla_b K = -\dot{K}^2 + \hat{K}^2. \quad (75)$$

Now using (51) and (52) in (75) we get

$$\nabla_a K \nabla^a K = \left(\frac{2}{3}\Theta - \Sigma + \phi \right) \left(\frac{2}{3}\Theta - \Sigma - \phi \right). \quad (76)$$

Hence for the 2-sheets intersecting the horizon (apparent or cosmological), the gradient of their Gaussian curvature is null. \square

As we are considering the scenario of gravitational collapse of massive stars, henceforth we will only concentrate on the apparent horizon. We have already seen that the curve

$$\Psi \equiv \frac{2}{3}\Theta - \Sigma + \phi = 0, \quad (77)$$

describes the apparent horizon. Let the vector $\Psi^a = \alpha u^a + \beta e^a$ be the tangent to the curve in the local $[u, e]$ plane. Then we must have $\Psi^a \nabla_a \Psi = 0$. Since we know that $\nabla_a \Psi = -\dot{\Psi} u_a + \hat{\Psi} e_a$, we can immediately see the slope of the tangent to the apparent horizon on the local $[u, e]$ plane is given by $\frac{\alpha}{\beta} = -\frac{\dot{\Psi}}{\hat{\Psi}}$. Now using this decomposition with the field equations (40) to (48), we obtain

$$\begin{aligned}\nabla_a \Psi &= \left(\frac{1}{3}\mu + p - \mathcal{E} + \frac{1}{2}\Pi - Q \right) u_a \\ &\quad + \left(-\frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E} + Q \right) e_a,\end{aligned}\quad (78)$$

and hence

$$\frac{\alpha}{\beta} = \frac{\frac{2}{3}\mu + \frac{1}{2}\Pi + \mathcal{E} - Q}{-\frac{1}{3}\mu - p + \mathcal{E} - \frac{1}{2}\Pi + Q}. \quad (79)$$

It is interesting to note that the matter thermodynamic quantities together with the Weyl scalar completely determine the tangent to the apparent horizon. We will define the apparent horizon to be locally *outgoing* at a point $p \in [u, e]$, if the slope of the tangent to the horizon is positive at p , that is $\frac{\alpha}{\beta} > 0$. Let the point p be labelled by the values of the local coordinates (t_0, χ_0) which are the affine parameters along the integral curves of u^a and e^a respectively. Then a locally outgoing apparent horizon at p would imply that the 2-sheets (spherical shell) labelled by $\chi_0 + \epsilon$ will get trapped later than $t = t_0$, while the 2-sheet labelled by χ_0 gets trapped at $t = t_0$. Finally $\frac{\alpha^2}{\beta^2} > (<) 1$ denotes the horizon to be locally timelike (spacelike). If $\frac{\alpha^2}{\beta^2} = 1$ then the horizon is null.

As an example let us consider the spherically symmetric vacuum spacetime. Then by Birkhoff's theorem the spacetime is static and hence $\Theta = \Sigma = 0$ [17]. Thus the horizon is described by the curve $\phi = 0$. In this case all the matter variables vanish, we have $\frac{\alpha}{\beta} = 1$ and we can easily see that the horizon is outgoing null. This is the *event horizon* of the Schwarzschild spacetime. Indeed if we calculate ϕ in Schwarzschild coordinates we get

$$\phi = \frac{2}{r} \sqrt{1 - \frac{2m}{r}}, \quad (80)$$

and $\phi = 0$ corresponds to the event horizon at $r = 2m$.

VI. END STATE OF A SPHERICAL GRAVITATIONAL COLLAPSE

Having derived the equations that govern the dynamics of the apparent horizon in a spherically symmetric spacetime, we are now in a position to analyse the end state of continual gravitational collapse. Let us consider the continual collapse of a general matter cloud to a final shell-focusing singularity, where all matter shells collapse to a zero physical radius. In particular, we analyse specifically the nature of the central singularity in detail to determine when it will be covered by the horizon, and when it will be visible and causally connected to outside observers. If there are future directed families of nonspacelike curves coming out from the singularity and reaching faraway observers, then the singularity will be naked. The absence of such families will give the covered case when the result is a black hole. We specifically focus on the central singularity as it has been shown a numerous times that if all the physically reasonable energy conditions are satisfied by the collapsing matter, then the non-central singularities are always covered [5].

Broadly, it can be stated that, if the neighbourhood of the centre gets trapped earlier than the singularity, then it is covered, otherwise it is naked with families of escaping nonspacelike future directed trajectories escaping away from it. Here we implicitly assume that the singularity curve (time taken for a spherical shell to become

singular) is a non-decreasing function of the affine parameter of the integral curve of the vector e^a . Otherwise non-central shells will become singular before the central shell and we will have to be contents with pathologies like shell crossing singularities.

Proposition 3. *Consider the continued collapse of a general spherically symmetric matter cloud from a regular initial epoch and obeying the physically reasonable energy conditions. If the following conditions are satisfied:*

1. *The spacetime is free of shell crossing singularities,*
2. *Closed trapped surfaces exist,*

then the necessary and sufficient condition for the central singularity to be locally naked is that the slope of the tangent to the apparent horizon at the central singularity is positive and non-spacelike ($\frac{\alpha}{\beta} \geq 1$).

Proof. Let the central singularity be denoted by $(t = t_{s0}, \chi = 0)$ in the $[u, e]$ plane. If the slope of the apparent horizon at the central singularity is greater than or equal to unity, and there are no shell crossings as assumed, then there exists an infinitesimal open set \mathcal{P} in the $[u, e]$ plane, defined by (for arbitrarily small positive numbers δ and ϵ)

$$\mathcal{P} = \{(t, \chi) | t_{s0} - \delta < t < t_{s0}, 0 < \chi < \epsilon\}, \quad (81)$$

which is outside the trapped region. The points $p \in \mathcal{P}$ are arbitrarily close to the central singularity and since they are not trapped, outgoing null geodesics ‘ γ ’ (with positive volume expansion) with their past end points at ‘ $p \in \mathcal{P}$ ’ are possible. Furthermore since the apparent horizon at the central singularity is locally outgoing and non-spacelike, there exists an open set $\mathcal{S} \subset J^+(\mathcal{P})$ (where J^+ denotes the causal future) in the $[u, e]$ plane where the volume expansion of the null geodesics ‘ γ ’, with their past end point in \mathcal{P} , continues to be positive. Hence the singularity is locally naked. The size of the open set \mathcal{S} depends on the scales of the problem (like mass or initial radius of the collapsing star) and one can always tune the scales to make the singularity globally naked (that is there may exist null geodesics reaching future null infinity with their past end point arbitrarily close to the central singularity) \square

This result is interesting as it transparently explains the role of the energy momentum tensor of the collapsing matter field as well as the Weyl curvature in making a spacetime singularity locally visible. Also, as shown in [19], if a null geodesic emerge from the singularity, then there exist families of future-directed nonspacelike curves which also necessarily escape from the same. The existence of such families is crucial to the physical visibility of the singularity. In the next proposition we show the crucial importance of the Weyl curvature in deforming the trapped region in such a way that the singularity becomes locally visible.

Proposition 4. *Consider the gravitational collapse of spherically symmetric perfect fluid obeying strong energy condition $\mu \geq 0$ and $\mu + 3p \geq 0$. If the following conditions are satisfied :*

1. *The spacetime is free of shell crossing singularities,*
2. *Closed trapped surfaces exist,*
3. *The central singularity is marginally naked ($\frac{\alpha}{\beta} = 1$),*

then the limit of $\frac{|\mathcal{E}|}{\mu+p}$ at the central singularity along the apparent horizon curve diverges.

Proof. We know that for a perfect fluid we have $Q = \Pi = 0$, and at the central singularity $\frac{\alpha}{\beta} = 1$ implies

$$\frac{\frac{2}{3}\mu + \mathcal{E}}{-\frac{1}{3}\mu - p + \mathcal{E}} = 1, \quad (82)$$

which can be simplified to

$$\left[\frac{\mathcal{E}}{\mu + p} - \frac{1}{3} \frac{\mu + 3p}{\mu + p} \right]^{-1} = 0. \quad (83)$$

For the perfect fluid satisfying the strong energy condition, $\frac{\mu+3p}{\mu+p}$ is finite and hence $\frac{|\mathcal{E}|}{\mu+p}$ at the central singularity along the apparent horizon tends to infinity. \square

The above result clearly shows that the electric part of the Weyl scalar (which is responsible for the tidal forces) must diverge faster than the energy density along the apparent horizon curve, for a singularity to be locally naked.

Corollary 1. *Consider the continued gravitational collapse of a spherically symmetric perfect fluid obeying the strong energy condition $\mu \geq 0$ and $\mu + 3p \geq 0$. If the spacetime is conformally flat then the end state of the collapse is necessarily a black hole.*

Proof. Conformally flat spacetime implies vanishing of the Weyl tensor. Hence we have $\mathcal{E} = 0$. Also for a perfect fluid $Q = \Pi = 0$. We therefore have

$$\frac{\alpha}{\beta} = \frac{-\frac{2}{3}\mu}{\frac{1}{3}\mu + p}. \quad (84)$$

Now the condition $\frac{\alpha}{\beta} \geq 1$ implies $\mu + p \leq 0$ which violates the strong energy condition. In fact one can explicitly calculate the norm of the tangent to show that

$$\Psi^a \Psi_a \propto -\frac{1}{3}(\mu + p)(\mu - 3p). \quad (85)$$

If the strong energy condition is satisfied we have $\mu + p > 0$, then we have the following cases:

1. If $\mu > 3p$ the Ψ^a is “ingoing” timelike.
2. If $\mu = 3p$ the Ψ^a is “ingoing” null.

3. If $\mu < 3p$ the Ψ^a is “ingoing” spacelike.

In all these cases the region around the centre gets trapped before the central singularity. Hence the singularity is always covered and the collapse end-state is always a black hole. \square

The above proposition highlights the importance of tidal forces in delaying the trapping. Absence of the Weyl tensor necessarily implies the absence of any tidal stresses, and we can easily see that the trapping occurs before the singularity formation.

VII. SOME SPECIFIC EXAMPLES

In this section we briefly discuss some of the well known examples of gravitational collapse scenarios in the light of the discussion in previous sections. As we will see below, in all these cases we can transparently determine the end state of the continued gravitational collapse using the formalism developed in this paper.

A. Oppenheimer-Snyder dust collapse

This was the first theoretical model of continued gravitational collapse, where the collapsing matter was assumed to be dustlike and homogeneous. In this case the interior metric is the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and is given by

$$ds^2 = -dt^2 + \frac{a(t)^2}{1 - kr^2} dr^2 + r^2 a(t)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (86)$$

The FLRW metric is conformally flat and hence $\mathcal{E} = 0$. Moreover, since the matter is dustlike we have $p = 0$. Hence the slope of the tangent to the central singularity, $\frac{\alpha}{\beta} = -2$. Thus the apparent horizon is ingoing timelike and the end state of the collapse is a black hole.

B. Lemaitre-Tolman-Bondi dust collapse

This is a well known gravitational collapse model where the Cosmic Censorship Conjecture is violated. Even though the collapsing matter is dustlike it may be inhomogeneous. The interior of the collapsing dust is described by the LTB metric

$$ds^2 = -dt^2 + \frac{R^2}{1 - r^2 b_0(r)} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (87)$$

Here $R(t, r)$ is the area radius of the collapsing dust shell and $b_0(r)$ denotes their energy profile. The system is specified by two free functions at the initial epoch, the energy profile $b_0(r)$ and the initial mass profile $F(r) \equiv r^3 \mathcal{M}(r)$. From the Einstein field equations we have

$$F' = \mu R^2 R'. \quad (88)$$

If we consider the marginally bound case where $b_0(r) = 0$, then the equation of motion of the collapsing shells are given by [20, 21]

$$\dot{R}^2 = \frac{F}{R}, \quad (89)$$

and the electric part of the Weyl scalar is [22]

$$\mathcal{E} = \frac{1}{3}\mu - \frac{r^3 \mathcal{M}(r)}{R^3}. \quad (90)$$

Following [20, 21], we can write $R = ra(r, t)$ where $a(r, t)$ is the ‘scale factor’ for a shell labelled ‘ r ’. Also we consider a smooth density profile at the centre and hence write the function $\mathcal{M}(r) \equiv \mathcal{M}_0 + \mathcal{M}_2 r^2$. We know that for the singularity curve to be an increasing function of ‘ r ’ to avoid shell crossings etc, we must have $\mathcal{M}_2 < 0$. Solving the equation of motion we get

$$a(r, t) = \left(1 - \sqrt{\mathcal{M}(r)t}\right)^{2/3}. \quad (91)$$

We can easily check (from Proposition 2 and Einstein’s equations) that the equation of the apparent horizon is given by $F = R$. Now the slope of the horizon is given by

$$\frac{\alpha}{\beta} = \frac{\frac{2}{3}\mu + \mathcal{E}}{-\frac{1}{3}\mu + \mathcal{E}}. \quad (92)$$

Calculating the slope at the central singularity (given by $t = t_{s_0} = 1/\sqrt{\mathcal{M}_0}$ and $r = 0$) and using (88, 91) we get

$$\frac{\alpha}{\beta} = \lim_{t \rightarrow t_{s_0}} \lim_{r \rightarrow 0} 1 - \frac{F'}{R'} = 1. \quad (93)$$

Hence we see that provided $\mathcal{M}_2 < 0$, the central singularity will be locally naked.

VIII. DISCUSSION

In this paper, working in a covariant and frame independent formalism, we successfully identified the physical and geometrical mechanisms responsible for delaying the trapped surface formation and making the central singularity locally naked during the continued gravitational collapse of a massive star. By working out the dynamics of the trapped region we transparently and quantitatively identified the role of Weyl curvature in deforming the trapped region in such a way that the singularity can be naked. As we know the Weyl curvature is responsible for the tidal force between nearby geodesics that generates the spacetime shear. In fact from the field equations for LRS-II spacetimes one can immediately see that the Weyl scalar is the source term for the shear evolution equation. Spacetime shear then deforms the apparent horizon and delays the trapping as shown in [6, 7].

These findings can have possible important observational signatures that can identify black holes from a

naked singularity, and hence observationally test the weak censorship hypothesis [23]. As we have seen, the Weyl curvature is the key feature that can generate a locally visible singularity. Moreover Weyl curvature is also the generator of gravitational waves [13]. Hence one can expect signatures of locally naked singularities from the gravitational waves radiated from a collapsing star.

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